1- Advanced Matrices

(1)Show that the following matrices are symmetric:

$$A=\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix},\ B=\begin{bmatrix}2 & i\\ i & 3i\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 3\\ 0 & 2 & 5\\ 3 & 5 & 4\end{bmatrix},\ D=\begin{bmatrix}1 & 0 & 3\\ 0 & 0 & -4\\ 3 & -4 & 8\end{bmatrix}$$

Also, find the eigenvalues of A and show that its eigenvectors are orthogonal.

(2)Determine the symmetric and skew symmetric matrices among the following:

$$A=\begin{bmatrix}3 & 4\\ 4 & 2\end{bmatrix},\ B=\begin{bmatrix}2 & 2i\\ 2i & 3\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 3\\ 2 & 2 & 5\\ 3 & 5 & 4\end{bmatrix},\ D=\begin{bmatrix}1 & 0 & 3\\ 0 & 3 & 4\\ 3 & 4 & 1\end{bmatrix}$$

$$E=\begin{bmatrix}0 & 2 & -3\\ -2 & 0 & 4\\ 3 & -4 & 0\end{bmatrix},\ F=\begin{bmatrix}0 & -2 & 3\\ 2 & 0 & 1\\ 3 & -1 & 0\end{bmatrix},\ G=\begin{bmatrix}1 & 4 & -2 & 2\\ 4 & 2 & 1 & 0\\ -2 & 1 & 3 & 1\end{bmatrix}$$

(3)Write each of the following matrices as a sum of two matrices, one is symmetric and the second is skew symmetric:

$$A=\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix},\ B=\begin{bmatrix}1 & 0 & 3\\ 2 & -1 & 5\\ 4 & -2 & 4\end{bmatrix},\ C=\begin{bmatrix}2 & 0 & 4\\ 0 & 3 & 4\\ 3 & 2 & 1\end{bmatrix}$$

(4)Determine the hermitain and skew hermitain matrices among the following:

$$A=\begin{bmatrix}3 & 4\\ 4 & 2\end{bmatrix},\ B=\begin{bmatrix}2 & 2i\\ 2i & 3\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 3\\ 2 & 2 & 5\\ 3 & 5 & 4\end{bmatrix},\ D=\begin{bmatrix}1 & 0 & 3i\\ 0 & 3 & 4\\ -3i & 4 & 1+i\end{bmatrix}$$

$$E=\begin{bmatrix}1 & 1+i & 3\\ -1+i & 2 & 0\\ 3 & 0 & 4\end{bmatrix},\ F=\begin{bmatrix}0 & 2i & 3\\ 2i & 0 & i\\ -3 & i & 0\end{bmatrix},\ G=\begin{bmatrix}0 & i & -2 & 2\\ i & 0 & -1 & 0\\ 2 & 1 & 0 & 1\end{bmatrix}$$

(5)Show that the following matrices are unitary:

$$A=\begin{bmatrix}-i & 0\\ i & i\end{bmatrix},\ B=\frac{1}{\sqrt{2}}\begin{bmatrix}i & -1\\ 1 & -i\end{bmatrix},\ C=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & -i\\ 0 & 1 & 0\\ i & 0 & -1\end{bmatrix}$$

(6)Show that the following matrices are orthogonal:

$$A=\frac{1}{2}\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix},\ B=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix},\ C=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{bmatrix}$$

$$D=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},\ E=\frac{1}{2}\begin{bmatrix}\sqrt{3} & -1 & 0\\ 1 & \sqrt{3} & 0\\ 0 & 0 & 2\end{bmatrix}$$

(7)Find the eigenvalues and the eigenvectors of each of the following matrices and write the diagonal form, if possible:

$$A=\begin{bmatrix}2 & 1\\ 0 & 4\end{bmatrix},\ B=\begin{bmatrix}1 & 2\\ -2 & -2\end{bmatrix},\ C=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix},\ D=\begin{bmatrix}2 & 1\\ 9 & 2\end{bmatrix},\ E=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$$

(8)Find the eigenvalues and eigenvectors of the following matrices :

$$A=\begin{bmatrix}3 & 2\\ 2 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & -2\\ -2 & 3\end{bmatrix},\ C=\begin{bmatrix}2 & 3\\ 3 & 2\end{bmatrix},\ D=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ E=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

(9)Find the eigenvalues and eigenvectors of the matrix and find $f(A)=\frac{A}{A^2-2I}$:

$$A=\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}$$ eigenvalues : 0, 2, 2

(10)Find the eigenvalues and eigenvectors of the matrix and find $f(A)=2^A$:

$$A=\begin{bmatrix}2 & 1 & 1\\ 2 & 1 & -2\\ -1 & 0 & -2\end{bmatrix}$$ eigenvalues : 3, – 1, – 1

(11)Find the eigenvalues and eigenvectors of the matrix and find $f(A)=\frac{4}{A+I}$:

$A = \begin{bmatrix} 8 & 5 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$ eigenvalues : 1, – 3, – 3

(12)Write the following expressions in matrix form and determine the type :

(a) $P = 3x^2 + 4y^2 + 2z^2 + 2xy - 2xz + yz$

(b) $P = 2xy + 4xz - 2yz - 3x^2 - 2y^2 - 2z^2$

(c) $P = 2xy + 6xz - 2yz + 3x^2 + y^2 + z^2$

(d) $P = 4x^2 + 3y^2 + 2z^2 + 2xy - 2xz + yz$

(e) $P = 4xy + 2xz - 2yz - 2x^2 - 3y^2 - 2z^2$

(f) $P = 2xy - 6xz + 2yz + x^2 + 2y^2 + 3z^2$

(13)Write the Hessian matrix of the functions:

(a) $f(x, y) = xe^y + 3y\cos x$

(b) $f(x, y) = y\ln x + x\sin y$.

(c) $f(x, y) = x + y + 3^{xy}$

(d) $f(x, y) = \tan^{-1} x + \tanh^{-1} y$.

(e) $f(x, y, z) = xye^z + \cos y + x^3 \ln z$

(f) $f(x, y, z) = ye^{2x} + \cos^3 y + x^3 \sin z^5$

(g) $f(x, y, z) = 2^{yz} + y\cosh x + x\sinh z$

2- Laplace Transformations

I-Find F(s) of each of the following functions:

(1) $f(t) = t^3 + 3t^2 - 2$

(2) $f(t) = \sqrt{t} + (2+t)t^2$

(3) $f(t) = t^2(2+t)^2$

(4) $f(t) = e^{3t} - t^2$

(5) $f(t) = \sin 2t + 3t^2 + \cosh t$

(6) $f(t) = e^{2t} + \cos 3t - 2\sinh t$

(7) $f(t) = sin^2 2t + cos^2 t$

(8) $f(t) = 3 + cos^3 2t$

(9) $f(t) = \sin 3t \,.\cos t$

(10) $f(t) = \sin 3t \,.\sin 2t$

(11) $f(t) = t^2.\sin 2t$

(12) $f(t) = 2 + e^{3t}\cos 4t$

(13) $f(t) = (t + \sin t)(3 - \cos 2t)$

(14) $f(t) = e^{-t}\sinh 3t$

(15) $f(t) = \sin 2t \,.\cosh 3t$

(16) $f(t) = \cos t \,.\sinh 2t$

(17) $f(t) = t + sinh^2 t$

(18) $f(t) = cosh^2 t.\sin 3t$

(19) $f(t) = t.e^{3t}.\cos 2t$

(20) $f(t) = e^{-t}.t.\sin 2t$

(21) $f(t) = 3^t.\cos 2t$

(22) $f(t) = 2^t.t - \sin 2t$

(23) $f(t) = (t-2)^4, \quad t > 2$

(24) $f(t) = \sin(t - \frac{\pi}{4}), \quad t > \frac{\pi}{4}$

(25) $f(t) = 2 + t, \quad t \geq 3$

(26) $f(t) = \sin(t - \frac{\pi}{4})$

(27) $f(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ t, & t > 1 \end{cases}$

(28) $f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ t^2, & t > 1 \end{cases}$

(29) $f(t) = \dfrac{\sin 2t}{t}$

(30) $f(t) = \dfrac{\cos 2t - \cos t}{t}$

(31) $f(t) = \dfrac{e^{-t} - e^{-2t}}{t}$

(32) $f(t) = \dfrac{\cos t - \cos 3t}{t}$

(33) $f(t) = \partial_0 + t$

(34) $f(t) = \partial_3.t$

(35) $f(t) = \partial_2.e^{3t}$

(36) $f(t) = U(t-3)$

II- Find the inverse Laplace transform of each of the following:

(1) $F(s) = 2 + \frac{3}{s}$ (2) $F(s) = \frac{3}{s-2} + \frac{1}{s^3}$ (3) $F(s) = \frac{1}{s^2-4} + \frac{3}{s^2+4}$

(4) $F(s) = \frac{s+1}{s^2+1}$ (5) $F(s) = \frac{1}{s+2} + \frac{1}{(s-2)^3}$ (6) $F(s) = \frac{s-4}{s^2-4} + \frac{3s}{s^2+4}$

(7) $F(s) = \frac{s}{s^2-3s+2}$ (8) $F(s) = \frac{s+4}{s^3-4s^2+3s}$ (9) $F(s) = \frac{3}{s^2-4s+4}$

(10) $F(s) = \frac{s}{s^2-2s+2}$ (11) $F(s) = \frac{1}{s(s-2)^3}$ (12) $F(s) = \frac{3}{s^2-4s-5}$

(13) $F(s) = \frac{2}{s(s^2+1)}$ (14) $F(s) = \frac{1}{s^2(s^2+1)}$ (15) $F(s) = \ln\frac{s+1}{s+2}$

(16) $F(s) = \tan^{-1} s$ (17) $F(s) = \tan^{-1}(1+s)$ (18) $F(s) = e^{-2s}$

(20) $F(s) = \frac{1}{s^2}e^{-3s}$ (20) $F(s) = \frac{s^2+1}{s^2+4}$ (21) $F(s) = \frac{s^2-4}{s^2-1}$

III- Solve the following differential equations:

(1) $y` - 2y = 0,$ $y(0) = 2$ $y = 2e^{2t}$

(2) $y`` + 4y` - 5y = 0,$ $y(0) = 1,\ y`(0) = 0$ $y = (\cosh 3t + \frac{2}{3}\sinh 3t)e^{-2t}$

(3) $y`` + 5y` + 6y = 0,$ $y(0) = y`(0) = 1$ $y = 4e^{-2t} - 3e^{-3t}$

(4) $y`` + 2y` + 5y = 0,$ $y(0) = y`(0) = 1$ $y = (\sin 2t + \cos 2t)e^{-t}$

(5) $y` + 3y = 1 + t,$ $y(0) = 0$ $y = \frac{2}{9} + \frac{1}{3}t - \frac{2}{9}e^{-3t}$

(6) $y`` + 4y` + 3y = e^{-t},$ $y(0) = y`(0) = 1$ $y = \frac{1}{2}te^{-t} + \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t}$

(7) $y`` - 3y` + 2y = 6e^{2t},$ $y(0) = y`(0) = 3$ $y = 9e^{t} - 6e^{2t} + 6te^{2t}$

3- Partial Differential Equations (PDE)

(1)Solve the following PDE :

(i) $u_x + 3u_y - 2u = 0$ (ii) $2u_x + 3u_y - 4u = 0$

(iii) $u_x - u_y = 4u$ (iv) $2u_x + u_y - 3x = 0$

(v) $3u_x + 4u_y - 5u = 10y$ (vi) $4u_x + 3u_y - 10u = 5$

(2)Find the solution of each of the following PDE :

(i) $2u_x + 3u_y - u = 0, \quad u(x,0) = 4$

(ii) $3u_x - u_y + u = 0, \quad u(x,0) = 2e^{5x} + 3e^{-3x}$

(iii) $3u_x - u_y + u = 0, \quad u(0,y) = 3e^{5y}$

(iv) $2u_x + u_y + u = 0, \quad u(x,0) = 4e^{-2x}$

(3)Classify and solve the following PDE :

(i) $u_{xx} - 5u_{xy} + 6u_{yy} = 0$ (ii) $u_{xx} - 5u_{xy} + u_{yy} = e^{2x+3y}$

(iii) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$ (iv) $u_{xx} - 5u_{xy} + 6u_{yy} = e^{3x+y}$

(v) $u_{xx} - u_x - 2u = 0$ (vi) $u_{yy} - 5u_y + 6u = 0$

(vii) $u_{xx} - u_x - 2u = e^{3x}$ (viii) $u_{yy} - 4u_y + 4u = e^{2y}$

(iv) $u_{xx} - 4u_{xy} + 4u_{yy} = xy^2$ (x) $u_{xx} - 4u_{xy} = \cos(2x + 3y)$

(xi) $u_{xx} + 3u_{yy} = 3x + y^2$ (xii) $u_{xx} - 4u_{xy} = \sin(4x + y)$

(4)Solve the following PDE :

(i) $u_{xx} - u_{xy} - 2u_{yy} + 2u_x + u_y - 5u = 0$ (ii) $u_{xx} + 3u_{xy} + 2u_{yy} + u_x - u_y + 6u = 0$

(iii) $u_{xx} - 4u_{xy} + u_x - 4u_y - u = 0$ (iv) $u_{xx} - 9u_{yy} - 2u_y + 4u = 0$

(5)By Laplace transform, solve the following PDE :

(i) $u_x + u_t = x, \quad u(0,t) = 0,\ t \ge 0,\ u(x,0) = 0,\ x \ge 0.$

(ii) $u_{xx} + u_t = e^t, \quad u(0,t) = 0,\ t \ge 0,\ u(x,0) = 0,\ x \ge 0.$

(iii) $u_x + u_{tt} = t, \quad u(0,t) = 0,\ t \ge 0,\ u(x,0) = u_t(x,0) = 1,\ x \ge 0.$

(iv) $u_{xx} + u_{tt} = 1, \quad u(0,t) = 0,\ t \ge 0,\ u(x,0) = u_t(x,0) = 0,\ x \ge 0.$